NONLINEAR MODELS FOR THE FRACTAL EVOLUTION OF CITIES AND TOWNS

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Abstract: *Some of the main ideas of the fractal city theory are briefly reviewed, and their applicability is tested for the medium and small-size Romanian urban settlements. Particularly, the diffusion-limited aggregation with dendritic-like growth (modelling The Central Place Theory of Christaller and Beckman) was proved to be in disagreement with the urban area development. Instead, the diffusion-limited aggregation with correlated percolation and self-organized criticality mechanisms are found to fit well the urban perimeter growth. Finally, the streets of a small Romanian town (Roman) were found to display the statistical structure of a scale-free network. The last model allows us to simulate complex phenomena like epidemic/rumour propagation and to find the most efficient lines of the urban development.*

Keywords: nonlinear model, cities and towns evolution, fractals, urban development,

1. HISTORICAL FRAMEWORK

More than seven decades ago, Englewood Cliffs from New Jersey published a pioneering book by Christaller (1933), where several key questions were for the first time posed: "What type of dynamics describe the growing of the urban locations?" and, further: "Are there laws which determine the number, size and distribution of towns?" The Christaller's theory – the so-called *central places theory*, later developed by Beckmann (1968), describes the urban morphology in the terms of the Euclidean geometry. The main idea is that the urban development is structured around a *central business district.*

The applicability of the central places theory is drastically limited by the exclusive using of Euclidean varieties as lines and surfaces. The modelling of the urban perimeter in Euclidean terms leads to results in strong discrepancy with the empirical evidence, especially for the large towns and cities.In the second half of the XX century, B. Mandelbrot (1975) opened the door for a more realistic description of the natural and social phenomena by introducing the mathematical varieties with fractional dimensions, usually called *fractals*. In the fractal theories, the dimension has a higher degree of generality than in Euclidean it has.

Cities are large physical objects animated and driven by human behaviour. By far the most interesting and difficult questions about them are about how the two connect: exactly how is the physical city linked to the human city? The consequent question is: what are the consequences of the physical form of the city for its human form, that is the patterns and dynamics of the economic, social, cultural and cognitive life that goes on in the city.

Living cities have intrinsically fractal properties, in common with all living systems. The pressure to accommodate both the automobile and increased population growth led twentieth-century urbanists to impose anti-fractal geometrical typologies. The fractal properties of the traditional city were erased, with disastrous consequences for the urban functionality.

As it was already shown in literature (Salingaros, 2001), older, pre-modernist cities are fractal, because they work on all scales. Mediaeval cities are the most fractal on the smaller scales up to 1 km, whereas 19th century cities work better on larger scales. Urban typologies used throughout history up until the twentieth century lead automatically to a fractal structure. But, the urban morphology is a product of the particular transportation system laid down by the government when the city was initially built. Later modifications to the transportation system lead to changes in city structure.

A city's life comes from its connectivity (Salingaros, 1998). In the present paper we discuss the connective properties of several undirected graphs to gain some insight into how city life arises. The simplest and most studied network with undirected edges was introduced by Erdös and Rényi (ER model) (Erdös & Rényi, 1959). In this network:

(i) the total number of vertices, *N*, is fixed;

(ii) the probability that two arbitrary vertices are connected equals *p*.

One sees that, on average, the network contains $pN(N-1)/2$ edges. The degree distribution is binomial:

$$P(k) = C_k^{N-1} p^k (1-p)^{N-1-k} \quad (1)$$

so the average degree is:

$$\bar{k} = p(N-1)$$

For large *N*, the distribution described in Eq. (1) takes the Poisson form:

$$P(k) = \frac{\exp(-\bar{k})\bar{k}^k}{k!}$$

Therefore, the distribution rapidly decreases at large degrees. Such distributions are characteristic for classical random networks (Dorogovtsev & Mendes, 2002).

As we shall see in the next section, the growing networks often self-organize into scale-free structures. A little change of parameters controlling their growth removes them from the class of scale-free structures. This is typical for the general self-organized criticality phenomena. Reasons for power-law distributions occurring in various systems, including urban systems, were a matter of interest of numerous empirical and theoretical studies. The models of networks growing with preferential linking can be reduced to the sandpile problem: at each increment of time, *m* new particles are distributed between the increasing number (by one per time step) of boxes according to some rule. Here, the boxes play the role of vertices. The particles are associated with edges. The probability that a new particle gets to a particular box depends on the filling of this box and on the filling numbers of all other boxes.

In Subsection 2.1 we simulate the urban growth by various models. We find that the sandpile model and Diffusion-Limited Aggregation (DLA model) with correlated percolation may fit the urban area of a small town. The problem of urban streets network is analysed in Subsection 2.2. Following some statistical data, we find that for a small town, the scale free network (simulated by Barabasi-Albert model of preferential attachment) fits better the data than the exponential (random) network.

The growth of a human settlement is essentially a self-organized process. A connection between self-organized criticality and urban growth may be performed by formulating an old sociological model (the Simon model) for networks in terms of vertices and edges (Bernhold & Ebel, 2001). This approach is based on two steps:

(i) At each increment of time, a new edge is added to the network.

(ii) a) Also, with probability *p* a new vertex is added, and the target end of the new edge is attached to the vertex.

b) With the complementary probability $1 - p$, the target end of the new edge is attached to the target end of a randomly chosen old edge.

In fact, both the original Simon model and the preferential linking concept are based on a quite general principle: *popularity is attractive*. Popular objects attract more new fans than the unpopular ones.

2. RESULTS

2.1 Numerical simulations of the growth process

The simulation follows the slightly modified cellular-automata model (Bak et al. 1987, Pica Ciamarra and Coniglio, 2006): the model is defined on a lattice, which we take for simplicity to be the two dimensional square lattice. There is a positive integer variable at each site of the lattice, called the height of the sand pile at that site. The system evolves in discrete time. In the first version (Fig. 1), all the grains are initially contented into the central site; then, the grains are added to the neighbouring sites with a power-law decay probability. In the second version (Fig. 2), one starts from a uniform distribution of heights. At each time step a site is picked randomly, and its height z_i is increased by unity. If the site height is larger than a critical value z_c, the site relaxes by toppling whereby z_c grains leave the site, and each of the four neighbouring sites gets $z_c/4$ grains. In case of toppling at a site at the boundary of the lattice, grains falling "outside" the lattice are *not* removed from the system, but they are added randomly to the highest ones. This process continues until all sites are stable.

As shown, in the DLA model, only a large central place or large cluster is generated (Fig. 1). The classical DLA model begins with an initial green "seed" in the center of the world. The particles move around the world randomly. When a particle hits a green square, it "sticks" and turns green (and a new particle is created to keep the process going). However, a real urban area is rather composed of central places that are spatially distributed following a certain hierarchy, thus the sand pile model (Fig. 4) offer a more realistic description of the urban perimeter (Fig. 3).

Such a structure may be also simulated by DLA with correlated percolation (Makse et al., 1995, 1998). Like the main DLA model, this model demonstrates diffusion-limited aggregation, in which particles moving (diffusing) in random trajectories stick together (aggregate) to form beautiful treelike branching fractal structures. There are many patterns found in nature that resemble the patterns produced by this model: crystals, coral, fungi, lightning, and so on (Witten Jr. and Sander, 1983)

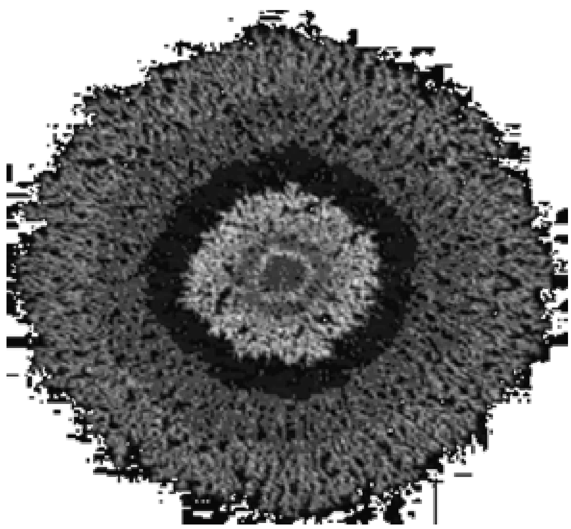
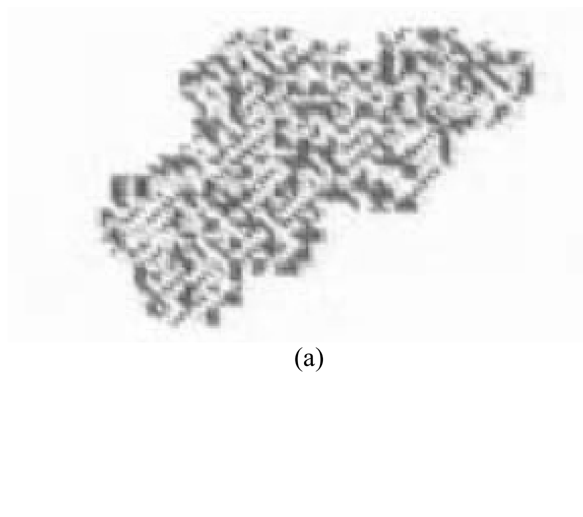
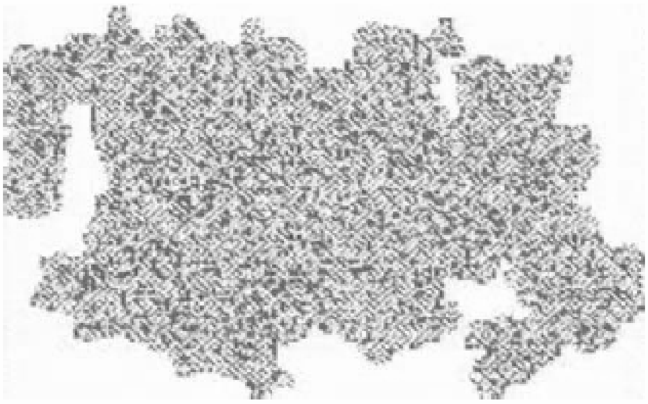


Figure 1 A numerical simulation of a growth process in a dendritic-like structure, from the DLA model. The growth begins in the centre and extends to the periphery



(a)



(b)

Figure 2 The result of the numerical simulation of a growth process in the sand pile model after (a) $n = 10^2$; (b) $n = 10^3$ simulation time steps.

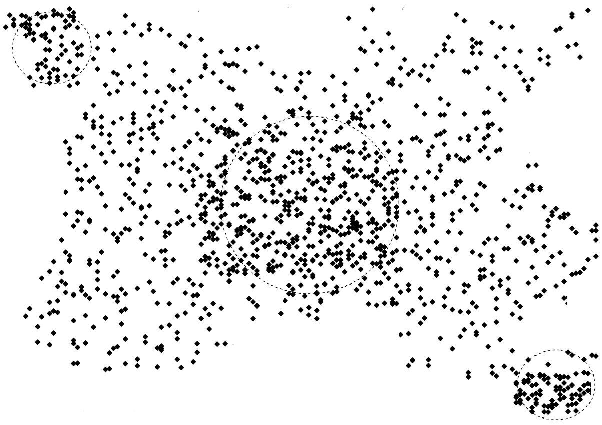


Figure 3 Structure of Roman in 2012. The residential and economic unit coordinates are obtained by dividing the map in 250 × 250 screen squares (See Ref. Roman – Interactive map). The surrounded zones are the ones at which the centrality indices (see Appendix) have the largest values.

Figure 4 The urban growth model simulated by DLA model with correlated percolation. The image is provided by a simulation using Netlogo applet (Wilensky, 2006) with the parameters: wiggle-angle: 60°; number of particles: 2500; ticks: 1221.

2.2 The streets of the town.Exponential or scale-free network?

The network metaphor in the analysis of urban and territorial cases has a long tradition especially in transportation /land-use planning and economic geography. All the previous approaches – though under different terms like "accessibility", "proximity", "integration", "cost", "connectivity", or "effort" – focus on the idea that some places (or streets) are more important than others because they are more central (Latora and Marchiori, 2004). The study of centrality in complex systems, however, originated in other scientific areas, namely in structural sociology, well before its use in urban studies; moreover, as a structural property of the system, centrality has never been extensively investigated metrically in geographic networks as it has been topologically in a wide range of other relational networks like social, biological or technological (Crucitti et al., 2006).

Let us discuss the simplest random network in which the number of vertices grows (Barabasi and Albert, 1999; Barabasi, Albert and Jeong, 1999). At each increment of time, let a new vertex be added to the network. It connects to a randomly chosen (i.e., without any preference) old vertex. Let connections be undirected, although it is inessential here. The growth begins from the configuration consisting of two connected vertices at time $t = 1$, so, at time t, the network consists of $t + 1$ vertices and t edges. The total degree equals $2t$. One can check that the average shortest-path length in this network is $\bar{\ell} \propto \ln t$ like in classical random graphs.

It is easy to obtain the degree distribution for such a network. We may label vertices by their birth times, $s =$ 0; 1; 2; ... ; t. Let us introduce the probability, $p(k; s; t)$, that a vertex s has degree k at time t. The master equation describing the evolution of the degree distribution of individual vertices is:

$$p(k,s,t+1) = \frac{1}{t+1} p(k-1,s,t) + \left(1 - \frac{1}{t+1}\right) p(k,s,t)$$

with $p(k, s = 0, t = 1) = \delta_{k,1}$, $\delta(k, s = t, t \geq 1) = \delta_{k,1}$.

This accounts for two possibilities for a vertex s:

(i) With probability $1/(t+1)$, it may get an extra edge from the new vertex and increase its own degree by 1.

(ii) With the complimentary probability $1 - 1/(t + 1)$, the vertex s may remain in the former state with the former degree. Notice that the second condition above makes the master equation non-trivial.

The total degree distribution of the entire network is:

$$P(k,t) = \frac{1}{t+1} \sum_{s=0}^{t} p(k,s,t)$$

Using this definition and applying $\sum_{s=0}^{t}$ to both sides of above master equation, we get the master equation for the total degree distribution:

$$(t+1)P(k,t+1) - tP(k,t) = P(k-1,t) - P(k,t) + \delta_{k,1}$$

The corresponding stationary equation (i.e. at $t \to \infty$) takes the form: $2P(k) - P(k-1) = \delta_{k,1}$

It has the solution of an exponential form:

$$P(k) \propto 2^{-k}$$

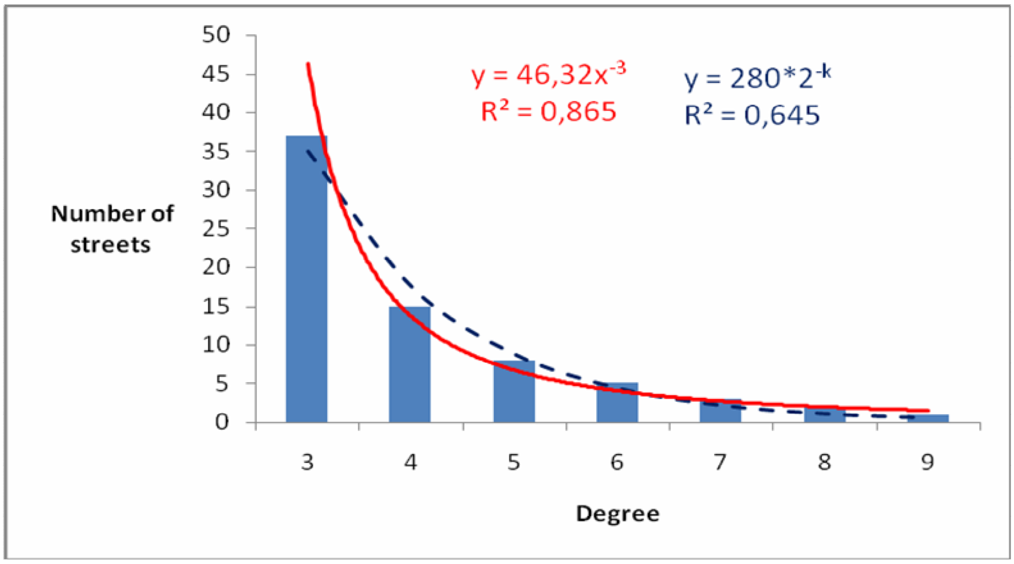


Figure 5 The distribution of degrees for $N = 71$ vertices in Roman (histogram). The continuous line fits to power distribution (scale-free network) while the dashed line fits to the exponential (random growing network). The Pearson coefficients RSQ are given in inset (See Ref. Roman – Interactive map).

At least several important large growing networks in nature are scale-free, i.e., their degree distributions are of a power-law form. The natural question is how they self-organize into scale-free structures while growing. What is the mechanism responsible for such self-organization? For explanation of these phenomena, the idea of preferential linking (preferential attachment of edges to vertices) has been proposed (Barabasi and Albert, 1999; Barabasi, Albert and Jeong, 1999).

We have demonstrated above that if new connections in a growing network appear between vertices chosen without any preference, e.g., between new vertices and randomly chosen old ones, the degree distribution is exponential. Nevertheless, in real networks, linking is very often preferential.

We describe here the simplest situation: The probability that the edge is attached to an old vertex is proportional to the degree of this old vertex, i.e., to the total number of its connections. At time t, the total number of edges is t, and the total degree equals $2t$. Hence, this probability equals $k/(2t)$. One should emphasize that this is only a particular form of a preference function.

For the BA model, the master equation takes the following form:

$$p(k,s,t+1) = \frac{k-1}{2t} p(k-1,s,t) + \left(1 - \frac{k}{2t}\right) p(k,s,t)$$

with the initial condition $p(k, s=0, t=1) = \delta_{k,1}$ and the boundary condition $p(k,t,t) = \delta_{k,1}$.

The master equation for total degree distribution:

$$(t+1)P(k,t+1) - tP(k,t) = \frac{1}{2}\left[(k-1)P(k-1,t) - kP(k,t)\right] + \delta_{k,1}$$

and, in the limit $t \to \infty$, the equation for the stationary distribution:

$$P(k) + \frac{1}{2}\left[kP(k) - (k-1)P(k-1)\right] = \delta_{k,1}$$

In the continuum k limit, this equation is of the form:

$$P(k) + \frac{1}{2}\frac{d\left[kP(k)\right]}{dk} = 0$$

The solution of the last equation is $P(k) \propto k^{-3}$.

Thus, the preferential linking provides a scale-free network and the exponent of degree distribution is –3.

An empirical study was performed on $N = 71$ nodes in Roman. The results are shown in Fig. 7. One can easily see that the urban streets structure is fitted better by the scale-free network than the exponential network. In Fig. 8 the urban streets structure is simulated by preferential attachment mechanism.

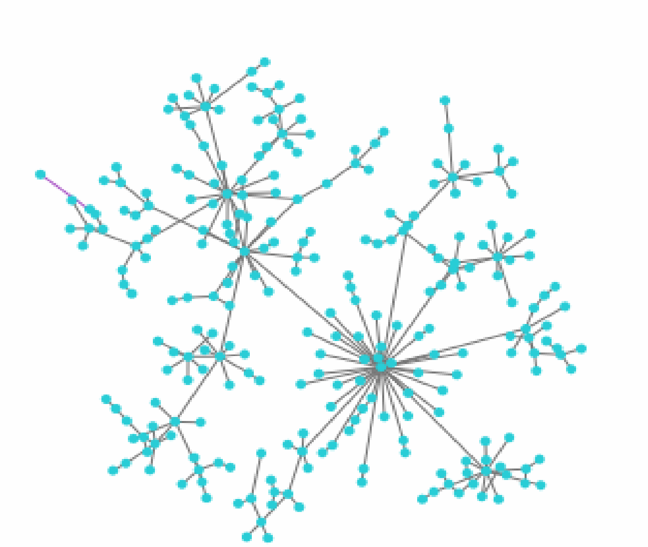


Figure 6 The urban streets structure simulated by preferential attachment mechanism. The image is provided by a simulation using Netlogo applet with the parameters: number of nodes: 142; ticks: 146.

3. CONCLUSIONS

In the present paper some basic ideas of the fractal city theory have been briefly reviewed. The universality of intrinsic fractality has been proved once again using particular statistical data.

Particularly the question of urban perimeter growth was pointed out. While an increasing amount of literature is devoted to the large cities structure and distribution, a relatively low interest has been so far given in the study of medium and small-size urban locations, especially those situated in the developing countries. Generally here are not mega-polis-like cities and the most urban centres are formed by merging some small units (villages). We found that self-organized criticality (the sand pile model) fits the urban perimeter better than the classical Central Place Theory. Nonetheless, the best fit of urban area has been performed by means of the Diffusion-Limited Aggregation with Correlated Percolation (DLACP) model.

Finally, the urban streets network was modelled starting from two models: the growing exponential network and the scale-free network. The last was simulated by means of Albert-Barabasi mechanism of preferential linking and it seems to fit better the empirical data than the first model, at least for small towns.

The study of urban perimeter growth and urban streets network may be useful in order to identify the most „central" points of the towns as well as to predict the further development of the towns in terms of cost and efficiency. Knowing the structure of the network we can analyse, also, complex phenomena such as the epidemic contamination spreading, and the fashion/fear/rumour propagation.

APPENDIX: Various centrality indices

Let us consider now N = the number of nodes/vertices in the streets network of the town. We consider a link as a street starting from an intersection situated next to the vertex.

a) *Degree and closeness centrality*

Degree centrality is based on the idea that important nodes have the largest number of ties to other nodes in the graph. The *degree* of a node is, as previously mentioned, the number of edges incident with the node, i.e. the number of first neighbours of the node. Defining the degree of node i as:

$$k_i = \sum_{j \in N} a_{ij}$$

the *degree centrality* (C^D) of the node i can be defined as (Nieminen, 1974):

$$C_i^D = \frac{k_i}{N-1} = \frac{\sum_{j \in N} a_{ij}}{N-1}$$

The normalization adopted is such that C^D takes on values between 0 and 1, and is equal to one in the case in which a node is connected to all the other nodes of the graph.

The simplest notion of closeness is based on the concept of minimum distance or geodesic *dij*, that is, the smallest sum of the edges lengths throughout all the possible paths in the graph from i to j in a weighted graph, and reduces to the minimum number of edges traversed, in a topologic graph.

The *closeness centrality* of point i (Freeman, 1979) is:

$$C_i^C = L_i^{-1} = \frac{N-1}{\sum_{\substack{j \in N \\ j \neq i}} d_{ij}}$$

where *Li* is the average distance from actor i to all the other actors. C^C is to be used when measures based upon independence are desired. Such an index is particularly meaningful for connected graphs.

b) *Betweenness centrality*

Interactions between two non-adjacent points might depend on the other vertices, especially on those on the paths between the two. Therefore points in the middle can have a strategic control and influence on the others. The important idea at the base of this centrality index is that a vertex is central if it lies between many of the other vertices. This concept can be simply quantified by assuming that the communication travels just along geodesics. Namely, if *njk* is the number of geodesics linking the two vertices j and k, and *njk (i)* is the number of geodesics linking the two vertices j and k that contain point i, the *betweenness centrality* of actor i can be defined as (Freeman, 1979):

$$C_i^B = \frac{1}{(N-1)(N-2)} = \sum_{\substack{j,k \in N \\ j \neq k \\ j,k \neq i}} \frac{n_{jk}(i)}{n_{jk}}$$

The betweenness centrality of the node *i* takes on values between 0 and 1, and it reaches its maximum when node *i* falls on all geodesics.

c) *Efficiency and Straightness centrality*

Efficiency and straightness centralities originate from the idea that the efficiency in the communication between two nodes *i* and *j* is equal to the inverse of the shortest path length *dij* (Latora and Marchiori, 2001).
In particular, the efficiency centrality of node *i* is defined as:

$$C_i^E = \frac{\sum_{\substack{j \in N \\ j \neq i}} \frac{1}{d_{ij}}}{\sum_{\substack{j \in N \\ j \neq i}} \frac{1}{d_{ij}^{Eucl}}}$$

where d_{ij}^{Eucl} is the Euclidean distance between nodes *i* and *j* along a straight line.
The straightness centrality is a variant of the efficiency centrality that originates from a different normalization (Vragović et al. 2005). The straightness centrality of node *i* is defined as:

$$C_i^S = \frac{\sum_{\substack{j \in N \\ j \neq i}} \frac{d_{ij}^{Eucl}}{d_{ij}}}{N-1}$$

This measure captures to which extent the connecting route between nodes *i* and *j* deviates from the virtual straight route.

d) *The overlapping index*

Another measure of the two countries connection strength is the *overlapping coefficient* (Gligor and Ausloos, 2008) defined for an unweighted network as:

$$O_{ij} = \frac{K_{ij}(k_i + k_j)}{2(N-1)(N-2)}, \; i \neq j,$$

where N is the number of vertices, k_i and k_j are the degrees of the two considered nodes, and K_{ij} is the number of common neighbours. For an unweighted network, O_{ij} does not account the edge directly linking *i* and *j* but rather to what extent the two nodes "overlap" by means of their common neighbours.

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