FROM THE “SIX DEGREES OF SEPARATION” TO THE WEIGHTED “SMALL-WORLD” NETWORKS

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Abstract. *Some of the most significant points in the study of the “small world” (SW) effect are briefly reviewed in the first section of the paper, starting from the Milgram’s sociological experiment, the paradigm of the “six degrees of separation”, and the Watts and Strogatz’ model. Based on interviews and questionnaires we found that the pupils network, in a school with about 1,000 pupils is a SW network with a mean degree of separation between 2 and 3. The problem is important taking into account that the spread of news, jokes, fashions, rumour, as well as epidemics, all take place by contact between individuals, far faster over a social network in which the average degree of separation is small than it can over one in which the average degree is e.g. 25. The third section is theoretical. The statistical ensemble of networks with fixed number of vertices was constructed and analyzed. A probability has been assigned to each two-individual connection by random attachment mechanism, and the corresponding partition function was built. The basic thermodynamic quantities, namely entropy, free energy, average energy per link and thermal susceptibility have been defined using the partition function. The variation of the thermodynamic quantities have been investigated during a thinking process of network deconstruction, which consist of removing the vertices one by one, in decreasing and, respectively, increasing order of the overlapping coefficients. Some evidences for critical points have been found, the corresponding phase transitions being generated by removing several special vertices from the system.*

Keywords: *small-world network, minimal path length, clustering coefficient, phase transition*

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1. INTRODUCTION

1.1 *Six degrees of separation*

The United Nations’ Department of Economic and Social Affairs estimates that the population of the world exceeded seven billion people in October, 2011. Perhaps the world of human society has become even larger nowadays. Nonetheless, when the people meet in an unexpected way, they often claim: “How small the world is!”. We demonstrate below that in a certain sense they are right. Despite the enormous number of people on the planet, the structure of social networks – the map of who knows whom – allows us to be all very closely connected to one another.

One of the first quantitative studies of the structure of social networks was performed by Stanley Milgram, then at Harvard University [1]. He performed a simple experiment as follows. He took a number of letters addressed to a stockbroker acquaintance of his in Boston, Massachusetts, and distributed them to a random selection of people in Nebraska. (Evidently, he considered Nebraska to be enough far from Boston, both in geographic and social terms). His instructions were that the letters were to be sent to their addresses by passing them from person to person, and that, in addition, they could be passed only to someone whom the passer new on a first-name basis. Since it was not likely that the initial recipients of the letters were on a first-name basis with a Boston stockbroker, their best strategy was to pass their letter to someone whom they felt was nearer to the stockbroker in some social sense: perhaps someone they knew in the financial industry, or a friend in Massachusetts. A reasonable number of Milgram’s letters did reach their destination, and Milgram found that it had only taken an average of six steps for a letter to get from Nebraska to Boston. He concluded that six was therefore the average number of acquaintances separating the pairs of people involved, and conjectured that a similar separation might characterize the relationship of any two people in the entire world. This situation was described by the syntagm “six degrees of separation” [2], a phrase which has since passed into popular language of sociology and was intensively exploited in the talk-show industry [3-5].

Given the multiple sources of error in the Milgram’s experiment, one may suppose that the number six of the degrees is probably not a very accurate one. However, the general result, that *two randomly chosen human beings can be connected by only a short chain of intermediate acquaintances* has been subsequently verified, and is now widely accepted. In the jargon of the field this result is referred to as the *small-world effect*.

Passing over the fashionable account of the phenomenon, we point out that the problem is crucially important for communications. Most human interactions take place directly between individuals. The spread of news, rumours, jokes, and fashions all take place by contact between individuals. As well, various diseases (from simple flues to the HIV virus) can spread far faster in a small world network than in a network where the average degree of separation is, say, ten thousand. That is why, during the last decade, the structure of the social networks and the small-world effect were extensively studied in literature.

1.2 *The terminology used in the study of networks*

The networks (or *graphs*) are composed of *vertices* (or *nodes*) connected by *edges* (or *links*). The edges may be directed or undirected. Correspondingly, we get a *directed* or a *undirected* network. To define the distances in the network, we consider the lengths of all edges equal to one. Here we do not consider networks with unit loops (edges started and terminated at the same vertex) and multiple edges, *i.e.*, we assume that only one edge may connect two vertices.

The structure of a network is described by its *adjacency matrix*, $\hat{A}$, whose elements consist of zeros and ones, for the *unweighted* graphs and numbers between 0 and 1 for the *weighted* networks. In the case of *unweighted* networks, an element of the adjacency matrix with undirected edges, a_{ij}, is 1 if vertices i and j are connected, and is 0 otherwise. Therefore, the adjacency matrix of a network with undirected edges is symmetrical. For a network with directed edges, an element of the adjacency matrix, a_{ij}, equals 1 if there is an edge from the vertex i to the vertex j, and equals 0 otherwise.

The degree of a vertex, k, is the total number of its connections. *In-degree*, k_i, is the number of incoming edges of a vertex. *Out-degree*, k_o is the number of its outgoing edges.

Hence, $k = k_i + k_o$. Degree is actually the number of nearest neighbors of a vertex, z_1. The network structure is given by the probability distributions: $P(k)$ = the degree distribution; $P_i(k_i) \equiv P(k_i)$ = in-degree distribution; $P_o(k_o) \equiv P(k_o)$ = out-degree distribution; $P(k_i, k_o)$ = the joint in- and out-degree distribution.

There are valid the following properties:

$$P(k) = \sum_{k_i} P(k_i, k - k_i) = \sum_{k_o} P(k - k_o, k_o)$$

$$P(k_i) = \sum_{k_o} P(k_i, k_o) \qquad (1)$$
$$P(k_o) = \sum_{k_i} P(k_i, k_o)$$

If a network has no connections with the exterior, then the average in- and out-degree are equal:

$$< k_i >= \sum_{k_i k_o} k_i P(k_i, k_o) =< k_o >= \sum_{k_i k_o} k_o P(k_i, k_o) \qquad (2)$$

Although the degree of a vertex is a local quantity, a degree distribution often determines some important *global* characteristics of random networks. Moreover, if statistical correlations between vertices are absent, $P(k_i; k_o)$ totally determines the structure of the network. One may define a "geodesic" distance between two vertices, i and j, of a graph with unit length edges. It is *the shortest-path length*, ℓ_{ij}, from the vertex i to the vertex j. If vertices are directed, ℓ_{ij} is not necessary equal to ℓ_{ji}. It is possible to introduce the distribution of the shortest-path lengths between pairs of vertices of a network and the average shortest-path length $<\ell> \equiv \ell$ of a network. The average here is over all pairs of vertices between which a path exists and over all realizations of a network.

The quantity ℓ determines the average distance between two nodes measured on the shortes path joining the two nodes. For a d-dimensional network containing N verices, one may demonstrate that $\ell \sim N^{1/d}$. In a fully connected network $\ell = 1$.

The average minimal path length, ℓ, may be roughly estimated for a network with random connections: if the average number of nearest neighbors of a vertex is z_1, then about $(z_1)^\ell$ nodes are placed at a distance ℓ from the vertex or closer. Hence, $N \sim (z_1)^\ell$ and one gets: $\ell \sim \ln N / \ln z_1$. We can see that the average minimal path length may have small values even for very large networks. This smallness expresses mathematically the small-world effect described in the previous section.

In order to describe the connections in the environment closest to a vertex, the so-called *clustering coefficient* is introduced. For a network with undirected edges, the number of all possible connections of the nearest neighbors of a vertex i (having $z_1^{(i)}$ nearest neighbors) equals to: $z_1^{(i)}[z_1^{(i)} - 1]/ 2$.

If only $y^{(i)}$ of them are present, the clustering coefficient of this vertex is:

$$C^{(i)} \equiv \frac{y^{(i)}}{z_1^{(i)}[z_1^{(i)} - 1]/2}$$

In other words, $C^{(i)}$ is the fraction of existing connections between nearest neighbors of the vertex. The physical meaning of the clustering coefficient is the probability that two nearest neighbors of a vertex are nearest neighbors also of one another. Averaging $C^{(i)}$ over all vertices of a network yields the clustering coefficient of the network, C. Remember that the notion of clustering was firstly introduced in sociology [6].

In a graph having all pairs of vertices connected (*fully connected network*), $C = 1$. In a graph having the vertices connected only to their first order neighbours (*tree-like network*), $C = 1$. In a classical random graph having N vertices, M edges, and an average number of firs order neighbours z_1 for each vertex, the following properies may be easily derived:

$M = z_1 N / 2$;

$$C = \frac{M}{N(N-1)/2} = \frac{z_1}{N-1}, \; 0 \le C \le 1. \; (3)$$

1.3 *The classical random networks*

The simplest and most studied network with undirected edges was introduced by P. Erdös and A. Rényi [7]. In their model the total number of vertices, N, is fixed and the probability that two arbitrary vertices are connected equals p.

This network contains, on average, $pN(N - 1)/2$ edges. The degree distribution is binomial:

$$P(k) = C_k^{N-1} p^k (1-p)^{N-1-k} \quad (4)$$

so that the average degree is $< k > = p(N - 1)$. For large values of N, eq. (4) has the form of the Poisson distribution:

$$P(k) = \frac{< k >^k}{k!} \exp(-< k >) \quad (5)$$

One can see that $P(k)$ decreases rapidly at large degrees k. In literature, this kind of graph is usually called *classical random network* [8].

However, there is a significant problem with the random graph as a model of social networks [9]. Let us consider a network of aquitances. The problem is that people's circles of acquaintances tend to overlap to a great extent. Your friend's friends are likely also to be your friends, or to put it another way, two of your friends are likely also to be friends with one another. This means that in a real social network it is not true to say that a person P has z^2 second neighbors, since many of those friends of friends are also themselves friends of person P. This property is the *clustering* of network described in the previous section by eq. (3).

A random graph does not show clustering. In a random graph the probability that two of person P's friends will be friends of one another is no greater than the probability that two randomly chosen people will be. On the other hand, clustering has been shown to exist in the social networks [10]. This is why the clustering coefficient C, was defined as the average fraction of pairs of neighbors of a node which are also neighbors of each other.

1.4 *The Watts-Strogatz model*

In § 1.2 we have shown that the random networks display the so-called "*small-world* effect" that consists in the fact that the averaged minimal path length is small even in the case of the large networks. Moreover, Watts şi Strogatz [11] pointed out another important property of the natural and social networks: in spite of the fact that the shortest path length is small – more exactly, of the order of $\log N$ – the clustering coefficient may display large values, much larger than the values corresponding to the random networks. The networks that are described by both above properties are called, in literature, "*small-world networks*". This kind of networks belongs to a transition class from ordered to disordered structures.

Obviously, this class of systems has interesting properties: they were constructed starting from *ordered* networks by various methods. The most common methods are the rewiring of links and the addition of random links between vertices. In this section we refer to the networks generated in these ways.

The original network of Watts and Strogatz is constructed in the following way:

- A regular one dimensional lattice with periodical boundary conditions is present. Each of L vertices has $z \ge 4$ nearest neighbors ($z = 2$ was not appropriate for Watts and Strogatz since, in this case, the clustering coefficient of the original regular lattice is zero).

• One takes all the edges of the lattice in turn and with probability p rewires to randomly chosen vertices. In such a way, a number of far connections appears. Obviously, when p is small, the situation has to be close to the original regular lattice. For large enough p, the network is similar to the classical random graph.

Watts and Strogatz studied the crossover between these two limits. The main interest was in the average minimal path length, ℓ, and the clustering coefficient (recall that each edge has unit length). The simple but exciting result was the following. Even for the small probability of rewiring, when the local properties of the network are still nearly the same as for the original regular lattice and the clustering coefficient does not differ essentially from its initial value, the average minimal path length is already of the order of the one for classical random graphs (see figure 1)

This result can be understood in an intuitive manner. In fact, the average minimal path length is very sensitive to the shortcuts. One can see that it is enough to make a few random rewirings to decrease ℓ by several times. On the other hand, several rewired edges cannot crucially change the local properties of the entire network. This means that the global properties of the network change strongly already at $pzL \sim 1$, when there is one shortcut in the network, i.e., at $p \sim 1/(Lz)$, when the local characteristics are still close to the regular lattice.

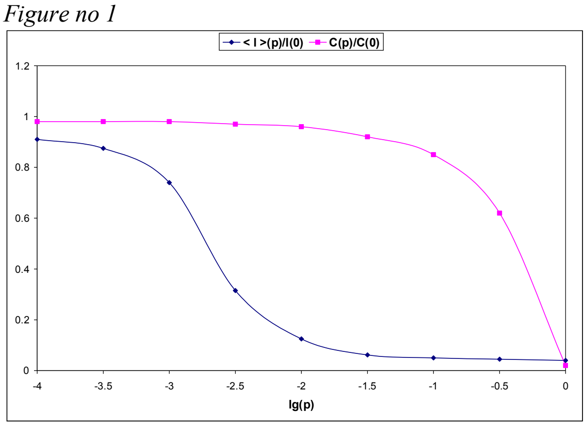
A large number of works focus on the distribution of diseases on such networks [12].

In Figure 2, the fraction of "infected" nodes in the network n_i / L is shown vs. time passed after some vertex was infected. At each time step, all the nearest neighbours of each infected vertex fall ill. At short times, $n_i/L \sim t^d$ but then, at longer times, it increases exponentially until the saturation at the level $n_i/L = 1$.

The Watts-Strogatz model and its variations seem exactly solvable analytically. Nevertheless, the only known exact result for the Watts-Strogatz model is its degree distribution. It was found to be a rapidly decreasing function of a Poisson kind [13].

Coefficient *C* in the Watts-Strogatz model versus the fraction *p* of the rewired links

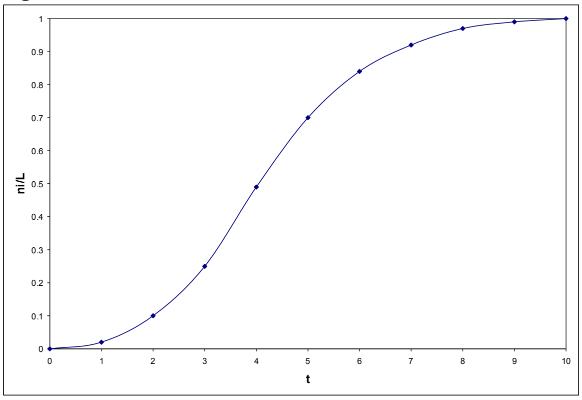
Figure no 1



The numerical simulation was performed using NetLogo soft. One can notice that C is practically constant in the range where ℓ decreases sharply.

Spreading of diseases in a "small-world" network

Figure no 2



The average fraction of infected nodes n_i/L is plotted vs. The elapsed time from the instant when the first vertex fell "ill". The numerical simulation was performed using NetLogo soft.

1.5 *Small-world solvable models*

The properties of the small-world networks may be studied on a simplified model that allows an analytic treatment. One starts from a 1-dimensional network composed of L nodes placed on a ring and chained by N links, each of them having the length equal to unit. In this case, the coordination number $z = 2$ and the clustering coefficient $C = 0$. We add a central vertex which connects to the initial nodes with a probability p by links having the length 1/2. More generally, we can add a number of extra vertices in the middle which are connected to a large number of sites on the main lattice, chosen at random (Figure 3). In fact, this model is similar to theWatts–Strogatz model in that the addition of the extra sites effectively introduces shortcuts between randomly chosen positions on the lattice, so it should not be surprising to find that this model does display the small-world effect.

Such nodes which have unusually high coordination numbers or which are linked to a widely distributed set of neighbours are frequently met in the real life. It seems that the "six degrees of separation" effect is due to a few people who are particularly well connected. We show below that even in the case where only one extra site is added, the model displays the small-world effect if that site is sufficiently highly connected ([8, 9]).

For the initial network $\ell_{(p=0)} = L/4$, and for completely connected network $\ell_{(p=1)} = 1$.

In Appendix we derive the distribution $P(\ell)$ of the minimal path lengths. At the limit $L \to \infty$ and

$p \to 0$, introducing the quantities $\rho \equiv pL$ (the average number of new added links) şi $z \equiv \ell / L$, the distribution is of the form:

$Q(z, \rho) \equiv$

$LP(\ell, p) = 2[1 + 2\rho z + 2\rho^2 z(1 - 2z)]\exp(-2\rho z)$ (6)

The distribution described by Eq. (7) is plotted in figure 5. In the same limit, the average minimal path length depends on the average number of new added connections as:

$$\frac{<\ell>}{L} \equiv <z> = \frac{1}{2\rho^2}[2\rho - 3 + (\rho + 3)\exp(-\rho)] \quad (7)$$

This function is plotted in figure 6. One can easily see that

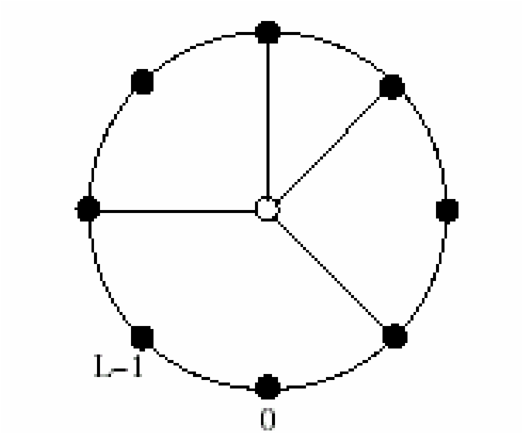
$<z>_{(\rho=0)} = 1/4$,

while

$< z >_{(\rho >> 1)} \to 1/\rho$, *i.e.* $\ell \to 1/p$.

A "small world" analytically solvable

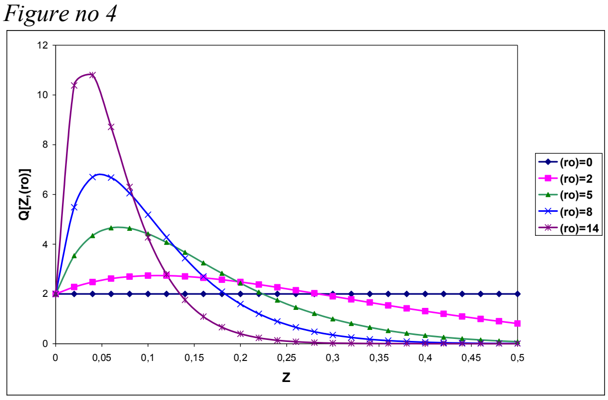
Figure no 3



The L vertices are situated on a ring and are connected by links having the length equal to unit. With probability p, some of these nodes are connected to the central vertex by links having the length equal to 1/2. This structure can model a real situation. The L families of a mountain village have few links among them except for the nearest neighbours. Nonetheless, some additional links may appear during the meetings of people at church [8].

The distribution $Q(z, \rho) = LP(l, p)$ of the normalized minimal path lengths $z \equiv \ell/L$ in a "small world" network (eq. (7))

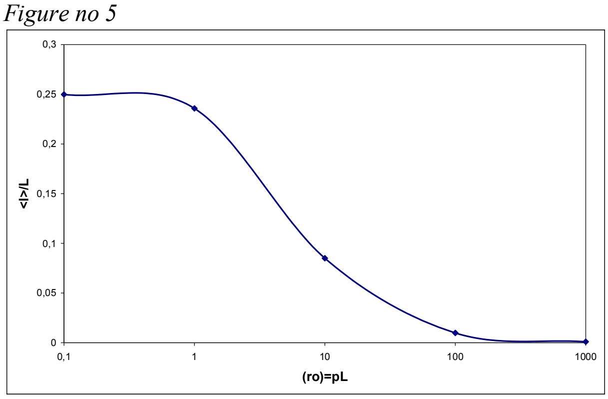
Figure no 4



Here L is the number of nodes of the network, while $\rho = pL$. We considered $L = 50$. The represented plots correspond to $\rho = 0$, 2, 5, 8, 11, 14.

The normalized minimal path length ℓ/L for a "small world" network, versus the number $\rho = pL$ of the new added links, in semi-logarithmic plot (eq. (8))

Figure no 5



2. EMPIRICAL RESULTS

We studied an acquaintance network composed of 40 pupils randomly chosen from various levels of study. We avoided to choose pupils from the same class and did not consider the links with teachers, focusing on the interaction among pupils.

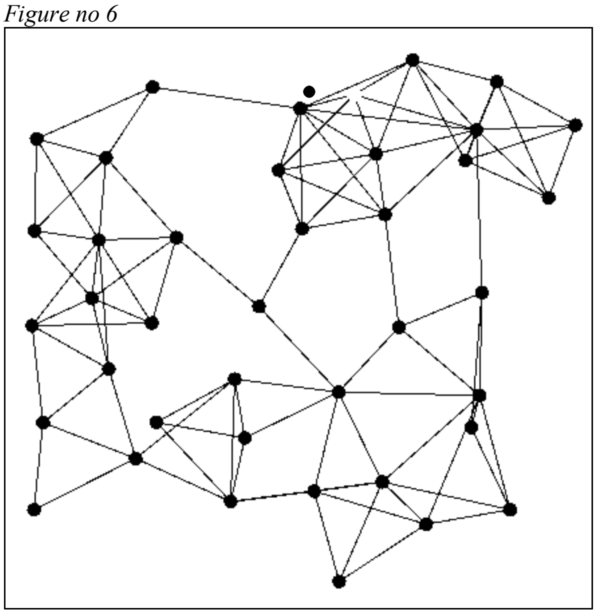
2.1 *Remaking Milgram's experiment*

A number of 10 tickets were given to 10 of the youngest pupils (11-13 y. o.) and 10 tickets were given to 10 of the oldest pupils (16-18 y. o.). The instruction was to hand on to addressee (whose name was written on the ticket) by person-to-person contacts. Each intermediate pupil added his own name/identification number to the ticket and gave it forward.

All the tickets reached to their addressees. The average number of steps was found about 2.7. Redoing their trajectories we found the structure of the network drawn in figure 6. The empirical clustering coefficient was found $C = 0.21$.

The structure of the pupils' network as it was obtained remaking the Milgram's experiment

Figure no 6



Further we used the NetLogo soft to generate a small-world network having the same characteristic values (ℓ and C). The corresponding small-world network was generated taking a probability of rewiring $p = 0.33$ (figures 7a and 7b).

Some results are synthesized in Table 1

Table 1: The number of nodes N, average degree of separation ℓ, and clustering coefficient C, for three real-world networks. The last column is the value which C would take in a random graph with the same size and coordination number.

Network	N	ℓ	C	C_{random}
pupils	40	2.7	0.21	0.05

Rewiring the links one-by-one we found that in the domain in which C decreases slowly from 0.29 to 0.21, ℓ decreases sharply from 5.38 to the final value 2.73.

2.2 *The weighted small-world network*

In the second part of the experiment we asked to each pupil implied in experiment to describe his/her friendship with the person whom he/she gave the ticket, by one of the variants:

i) Close friendship (3 points);

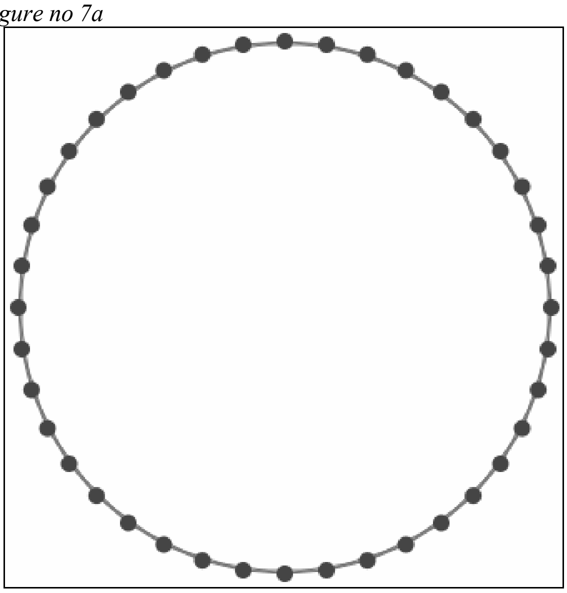
ii) Friendship (2 points)

iii) Casual acquaintance (1 point).

The total number of points was 276. In the empirical network, at the corresponding link we attached one of the the weights: 3/276, 2/276 or 1/276. In order to complete the network, we assign to the unrealised links the weight $w = 0$. In this way, a weighted network is obtained and its adjacency matrix is completely determined.

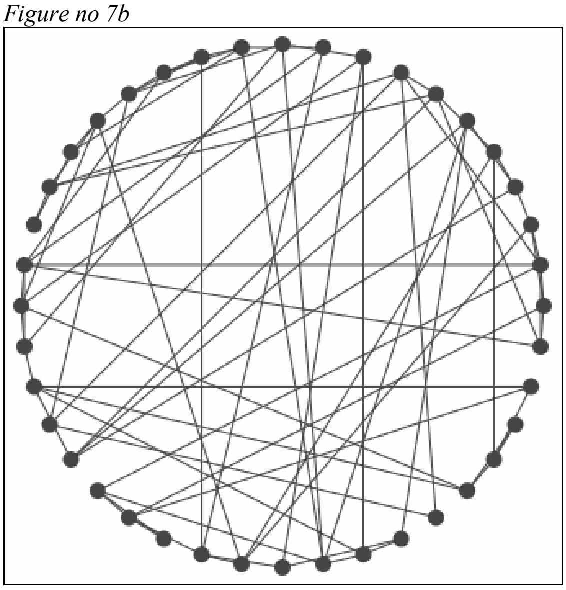
The initial regular network used for generating the small-world network by Watts-Strogatz method. The setting values: $C = 0.5$; $\ell = 5.38$; $p = 0.5$

Figure no 7a



The small-world network that simulates the pupils' network, generated by rewiring all the links. The characteristic values: $C = 0.2$; $\ell = 2.73$; $p = 0.33$

Figure no 7b



Now we consider the pupils as vertices of this fully connected weighted network, having attached to the edges their weights:

$w_{ij} = \hat{A}_{ij}$ (8)

fulfilling the relations:

i) $0 \leq w_{ij} \leq 1$; and

ii) $w_{ij} = w_{ji}$.

Another measure of the two vertices connection strength is the *overlapping coefficient* [10] defined for an unweighted network as:

$$O_{ij} = \frac{K_{ij}(k_i + k_j)}{2(N-1)(N-2)}, \; i \neq j, \quad (9)$$

where N is the number of vertices, k_i and k_j are the degrees of the two considered nodes, and K_{ij} is the number of common neighbours. For an unweighted network, O_{ij} does not account the edge directly linking i and j but rather to what extent the two nodes "overlap" by means of their common neighbors.

For a weighted network, Eq. (9) may be generalized as:

$$O_{ij} = \frac{1}{2(N-1)(N-2)} \sum_{\substack{l=1 \\ l \neq i,j}}^{N} (w_{il} + w_{jl}) \left(\sum_{\substack{p=1 \\ p \neq i}}^{N} w_{ip} + \sum_{\substack{q=1 \\ q \neq j}}^{N} w_{jq} \right) \quad i \neq j.$$

(10)

One can easily see that $0 < O_{ij} < 1$, and $O_{ij} = 1$ only for all $w_{ij} = 1$, *i.e.* fully connected non-weighted network. However, for a weighted network, O_{ij} can never be zero.

Summing all O_{ij}'s for one vertex, one gets an alternative measure of the *vertex strength*:

$$O_i = \sum_{j=1}^{N} O_{ij} \quad (11)$$

Table 2 The overlapping index of vertices in the weighted network of pupils. The vertices (pupils) are assigned an identification number from v01 to v40.

v01	12.56	v15	8.71	v29	8.07
v02	12.23	v16	8.65	v30	8.05
v03	11.97	v17	8.65	v31	7.94
v04	11.79	v18	8.60	v32	7.84
v05	10.56	v19	8.55	v33	7.73
v06	9.55	v20	8.52	v34	7.73
v07	9.47	v21	8.52	v35	7.59
v08	9.42	v22	8.47	v36	7.42
v09	9.34	v23	8.34	v37	7.11
v10	9.23	v24	8.23	v38	6.79
v11	9.15	v25	8.13	v39	6.79
v12	9.10	v26	8.10	v40	6.29
v13	8.94	v27	8.10		
v14	8.76	v28	8.07		

The results for the considered weighted network are shown in Table 2, in decreasing order of O_i.

3. PHASE TRANSITIONS IN THE WEIGHTED SMALL-WORLD NETWORK

3.1 *Statistical mechanics of the weighted networks*

In the same way as in Ref. [14] we can elaborate a sort of statistical mechanics of the weighted network. Firstly, we can try to find the probability of having the weight w_{ij} assigned to the edge i - j on the hypothesis that in the isomorphic multi-graph the links are attached *randomly* between the edges. If we have N vertices, the corresponding number of possible connections becomes:

$$\binom{N}{2} = N(N-1)/2$$

and the probability of having w_{ij} simple edges between the vertices (*i*) and (*j*) is read:

$$p_{ij} = C\frac{1}{\binom{N}{2}^{w_{ij}}} = C\left(\frac{N(N-1)}{2}\right)^{-w_{ij}}.$$

Introducing the notation: $\Lambda = N(N-1)/2$, after the normalization:

$$\sum_{\substack{i,j\\ i>j}} p_{ij} = 1$$

the above probability becomes:

$$p_{ij} = \frac{\Lambda^{-w_{ij}}}{\sum_{\substack{i,j\\ i>j}} \Lambda^{-w_{ij}}} \quad (12)$$

Finally, one can turn back to the initial network with $0 \le w_{ij} \le 1$; defining:

$$\beta = \ln\Lambda = \ln\frac{N(N-1)}{2}, \quad (13)$$

Eq. (12) gets the more familiar "canonical" form:

$$p_{ij} = \frac{\exp(-\beta w_{ij})}{\sum_{\substack{i,j\\ i>j}} \exp(-\beta w_{ij})}. \quad (14)$$

Note that the parameter β in Eq. (13) is not related to any temperature. Nonetheless, β can be seen as an *internal* parameter of the statistical ensemble of *N*-vertex networks, in the same way in which the temperature is for the canonical ensemble. Unlike the thermodynamic meaning, the changing of β does involve neither warming nor cooling process, but it simply means the shifting from a statistical ensemble to another one.

On the above assumptions, some basic thermodynamic quantities can be defined in correspondence to the classical statistical mechanics, as follows:

- The partition function:

$$Z = \sum_{\substack{i,j\\ i>j}} \exp(-\beta w_{ij}) \quad (15)$$

- The entropy:

$$S = -\sum_{\substack{i,j\\ i>j}} p_{ij} \ln p_{ij} =$$

$$= -\sum_{\substack{i,j\\ i>j}} \frac{\exp(-\beta w_{ij})}{\sum_{\substack{i,j\\ i>j}} \exp(-\beta w_{ij})} \ln \frac{\exp(-\beta w_{ij})}{\sum_{\substack{i,j\\ i>j}} \exp(-\beta w_{ij})} \quad (16)$$

- The free energy:

$$F = \frac{1}{\beta}\ln Z = \frac{1}{\beta}\ln\sum_{\substack{i,j\\ i>j}} \exp(-\beta w_{ij}) \quad (17)$$

- The average energy / link:

$$< w >= \sum_{\substack{i,j\\ i>j}} p_{ij} w_{ij} = \sum_{\substack{i,j\\ i>j}} \frac{w_{ij}\exp(-\beta w_{ij})}{\sum_{\substack{i,j\\ i>j}} \exp(-\beta w_{ij})} \quad (18)$$

- The "thermal" susceptibility:

$$\Lambda\chi_T = \frac{d < w >}{d(1/\beta)} = -\beta^2 \frac{d < w >}{d\beta} = \quad (19)$$
$$= \beta^2\left[< w^2 > - < w >^2\right]$$

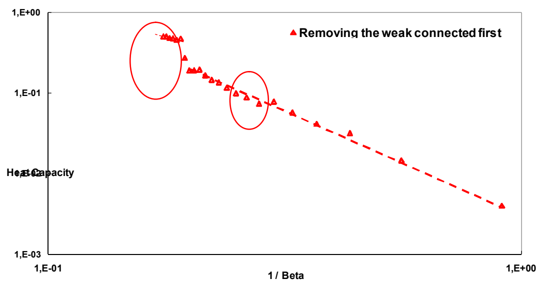
3.2 *Deconstructing the weighted network*

In order to get some more information about the structure of weighted network we examine it during a thinking process of decomposition, which consists in removing the vertices one by one, in decreasing and, respectively, increasing order of the overlapping coefficients from Table 2. Keeping somehow the "thermodynamic" analogy, the quantities defined by Eqs. 13-16 are studied as functions of β, which is a measure of the number of remainder vertices, and ($1 / \beta$), which is a measure of the number of removed vertices.

Some results are plotted in figures 8-9.

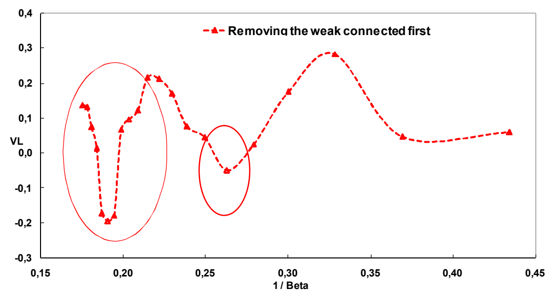
The thermal susceptibility variation in the network deconstruction process, in log-log plot. The weak connected nodes were removed first

Figure no 8a



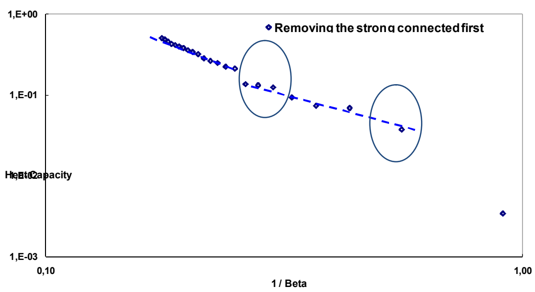
The fourth central cumulant V_L variation in the vicinity of the critical points marked out in figure 8a

Figure no 8b



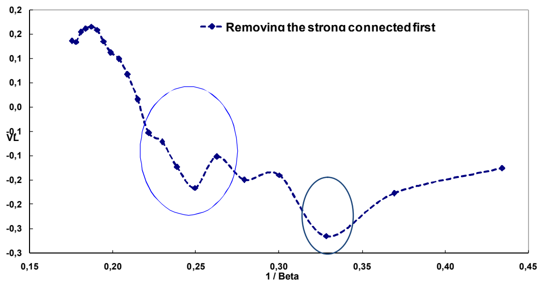
The thermal susceptibility variation in the network deconstruction process, in log-log plot. The strong connected nodes were removed first

Figure no 9a



The fourth central cumulant V_L variation in the vicinity of the critical points marked out in figure 9a

Figure no 9b



During the deconstruction process, the thermal susceptibility (the heat capacity) displays, by far, the most interesting behaviour. This quantity has two significant discontinuity points. We test these points by calculating the fourth central cumulant:

$$V_L = 1 - \frac{< w_{ij}^4 >}{3 < w_{ij}^2 >^2} \quad (20)$$

one finds that V_L has local minima at the values corresponding to the largest fluctuations of $\Lambda\chi_T$, This behavior indicates the presence of two critical points [15].

In literature, a first order phase transition was considered so far the transition from the regular lattice to the small-world network [16]. As well, phase transitions appear during the processes of epidemics spreading and percolation [17]. The behavior of the thermodynamic quantities defined above accredits a new kind of phase transitions that appear during the simulated process of the network deconstruction.

4. CONCLUSION

We reviewed briefly several of the most significant steps on the way from the Stanley Milgram's "six degrees of separation" to the modern approaches of the small-world networks.

As long as the most human interactions take place directly between individuals, we may expect that a large number of social networks to display the small-world effect. That is why, in the recent literature, the spread of rumours, information and diseases are mostly studied on small-world networks.

In the second section we studied the small-world effect on the particular network of pupils in the school. We found that, in spite of some unavoidable sources of error, any two randomly chosen pupils can be connected by only a short chain of intermediate acquaintances. Particularly, the investigated network may be easily simulated by Watts-Strogatz method of links rewiring.

One alternative to the Watts-Strogatz method is to explain the small-world effect by existence of few nodes in the network which have unusually high coordination numbers or which are linked to a widely distributed set of neighbours. An alternative model of this kind was firstly elaborated by S. N. Dorogovtsev and J.F.F. Mendes and was proved to be analitically solvable.

We proved this assumption in the last section. In order to approach better the real-life situation we attached some weights to the links, getting in this way a weighted small-world network. During the thinking process of network deconstruction, some thermodynamic quantities carry forth critical behaviour indicating phase transitions occurrence. The meaning behind this finding is the existence of several nodes whose removal leads to qualitative changes in the network structure. On this way we may conclude that, to a large extent, the small-world effect is due to a few people who are particularly well connected. Their role in the small-world network running and evolution remains a task to study in the future.

APPENDIX

The analytic solution for the "small world" model in § 1.5

Let us consider the network in figure 4 with directed edges. The case of undirected edges is also analytically solvable, in a similar way, but the mathematical work is more complicated. Nonetheless, the results do not differ qualitatively in the two cases.

In order to get the distribution $P(\ell)$ of minimal paths, an intermediate step is to infer a recurrence relation for $P(\ell, k)$ *i.e.* the probability that the minimal path length between two nodes be ℓ when the "Euclidian" distance between nodes (measured on the ring) is k. Obviously, the property $\sum_{l=1}^{k} P(l,k) = 1$ is fulfilled. We calculate the quantities $P(l \le k, k)$ for several small values of ℓ and k, and, starting from them we infer a general solution.

To find $P(\ell, k)$ for the model in figure 3 we have to take into account all the possible combinations of the edges connecting the center with the vertices i = 1, 2, …, k. To each of these edges corresponds a probability p (and a complementar probability $1–p$ the link be not realised). The lengths of edges between two adjacent nodes is 1, while the links joining these nodes to center is 1/2. For small values of ℓ and k we easily get:

$P(1, 1) = 1$;
$P(1, 2) = p^2$,
$P(2, 2) = 1 - p^2$;
$P(1, 3) = p^2$,
$P(2, 3) = 2p^2(1 - p)$,
$P(3, 3) = 1 - p^2 \cdot 1 - 2p^2(1 - p)$;
$P(1, 4) = p^2$,
$P(2, 4) = 2p^2(1 - p)^1$,
$P(3, 4) = 3p^2(1 - p)^2$
$P(3, 3) = 1 - p^2[1 \cdot (1 - p)^0 + 2(1 - p)^1 + 3(1 - p)^2]$; (A1)

……………….

$$P(\ell < k, k) = \ell p^2 (1 - p)^{\ell - 1},$$

$$P(\ell = k, k) = 1 - p^2 \sum_{i=0}^{k-1} i(1-p)^{i-1} \quad \text{(A2)}$$

The minimal paths distribution is:

$$P(\ell) = \frac{1}{L-1}\sum_{k=1}^{L-1} P(\ell,k) = \frac{1}{L-1}\sum_{k=\ell}^{L-1} P(\ell,k). \quad \text{(A3)}$$

Substituting (A2) in (A3) one gets:

$$P(\ell) = \frac{1}{L-1}[1 + (\ell - 1)p + \ell(L - 1 - \ell)p^2](1-p)^{\ell-1} \quad \text{(A4)}$$

The average minimal path length is:

$$< \ell > = \sum_{l=1}^{L-1} \ell P(\ell) \quad \text{(A5)}$$

For simplicity, we drop further the brackets, keeping the notation ℓ for the average minimal path length:

$$\ell = \frac{1}{L-1}\left[\frac{2-p}{p}L - \frac{3}{p^2} + \frac{2}{p} + \frac{(1-p)^n}{p}\left(n - 2 + \frac{3}{p}\right)\right] \quad \text{(A6)}$$

In order to get a description of the transitory regime between the regular lattice and the random network, we consider the limits $L \to \infty$ and $p \to 0$ under restriction that the quantities $\rho \equiv pL$ and $z = \ell/L$ are fixed. In these conditions, from eq. (A4) we get the continuous distribution $Q_{dir}(z, \rho)$:

$Q_{dir}(z, \rho) \equiv LP(l, p) = 1 + \rho z + \rho^2 z(1 - z)\exp(-\rho z)$ (A7)
where $0 \leq z \leq 1$.

Further, from eq. (A6) (or (A7)) we get the normalized average minimal path length:

$$\frac{\ell}{L} \equiv \langle z \rangle = \frac{1}{\rho^2}[2\rho - 3 + (\rho + 3)\exp(-\rho)] \quad \text{(A8)}$$

Eqs. (A7) şi (A8) are valid for the networks with directed edges. The relation (7) from §1.5, that is also valid for networks with undirected edges, can be derived from the eqs. (A7) and (A8) by means of the variable changes:
$z \to 2z$ and $Q(z, \rho) = 2\, Q_{dir}(2z, \rho)$.

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